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Thermoelastic Interaction Between a Hole and an Elastic Circular Inclusion

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The plane interaction problem for a circular elastic inclusion with an arbitrarily oriented hole, located either in the matrix or in the inclusion under a remote uniform heat flow, is solved. The proposed method is based on the complex variable theory and the existing solutions for dislocation functions, which permit us to formulate boundary integral equations with a weak singular kernel. The unknown coefficients left in the singular integral equations thus can be solved numerically by applying the appropriate interpolation formulas. The hoop stresses, which are directly related to the coefficients of dislocation functions, are then determined accordingly. Several numerical examples are given to demonstrate the use of the present approach. Comparisons between the calculated numerical results and the corresponding existing solutions show that the method proposed is effective, simple, and general.

Introduction

STUDIES on the interaction among multiple inclusions or inhomogeneities have received much attention in research communities. One of the most difficult parts of solving this problem is that the continuity conditions across the interfaces between the matrix and the inclusions are required to be satisfied. Exact closed-form solutions can be found only for a single inclusion embedded in an infinite isotropic or anisotropic medium,[1–3] whereas no exact closed-form solutions are available to the corresponding problem of multiple inclusions with two or more separate interfaces. The particular problem of two circular elastic inclusions was solved by Goree and Wilson,[4] who applied a mapping function in conjunction with a bilinear transformation. The same problem was resolved by Sendeckyj,[5] who used a successive approximation method. Zimmerman[6] further examined the stress concentration around two circular holes in an infinite plate solved by Kienzler and Duan,[7] who derived a simple formula to obtain the distribution of hoop stresses by applying the Fourier series expansion method. Based on the method of pseudotractions, Horii and Nemat-Nasser[8] studied the elastic problem of interacting inhomogeneities. Gong and Meguid[9] obtained a series solution for the problem containing any arbitrary number of inclusions by using the method of the Laurent series expansion.[10] Following the concept of the heterogenization technique,[11] Chao and Gao[12] found an explicit solution of elastostatics problems in heterogeneous solids.

All of the aforementioned studies have concentrated on multiple-inclusion problems under isothermal loading conditions. Very few solutions of the corresponding thermoelastic problem appear in the literature. Recently, Chao et al.[13] derived the general expressions of the complex potentials for the thermoelastic problem with multiple inclusions that satisfy the prescribed continuity conditions for each circular inclusion. The remaining unknown coefficients appearing in a system of coupled algebraic equations were solved by the perturbation technique. However, as the two neighboring inclusions become close to each other, the calculations become extremely cumbersome and less accurate. A precise knowledge of the stress field near interacting inclusions is extremely helpful in predicting failure initiation and damage tolerance in composite material systems. Therefore, a challenging problem is to study the interaction behavior among multiple inclusions under remotely applied loads.

In this paper, a unified approach is proposed to solve the problem with a hole of any arbitrary shape interacted with an elastic circular inclusion under remote uniform heat flow. Green's function for the inclusion problem when the singularity resides either inside the matrix or inside the inclusion was derived first, based on the method of analytical continuation. To satisfy the traction-free condition along the hole boundary, a system of singular integral equations is established by using the related Green's function in conjunction with the technique of superposition. In the following, the temperature and stress complex potentials are obtained for a dislocation located either inside the matrix or inside the inclusion. The hole contour is simulated as a polygon of N line segments using the appropriate interpolation formulas. The hoop stress along the hole boundary can be determined in terms of the unknown coefficients appearing in the interpolation formulas. Numerical examples associated with a circular hole interacting with the circular inclusion under a remote uniform heat flow are examined in detail. The problem when a hole is embedded in the inclusion is also discussed. The obtained results will be helpful in deepening understanding of the thermoelastic interaction behavior when an existing defect, such as a hole, and the surrounding inclusions become close to each other.

Temperature Field

Consider two homogeneous, isotropic elastic materials. Let one occupy the infinite region S_1, exterior to the circle of radius a and the other occupy the region S_2, interior to the circle of radius a (Fig. 1). The governing equations for two-dimensional steady-state heat conduction problems are given by

$$\nabla^2 T_j(x, y) = 0, \qquad j = 1, 2$$

with boundary conditions $Q_1 = Q_2$ and $T_1 = T_2$ along the interface $|z| = a$, where the resultant heat flux Q_j ($j = 1, 2$) and temperature T_j ($j = 1, 2$) for each medium are expressed in terms of the complex potential $g_j'(z)$ as

$$Q_j = \int (q_{xj}\,\mathrm{d}y - q_{yj}\,\mathrm{d}x) = -k_j\,\mathrm{Im}[g_j'(z)] \tag{1}$$

$$T_j = \mathrm{Re}[g_j'(z)] \tag{2}$$

where Re and Im are the real part and imaginary parts of the bracketed expression, respectively. The quantities q_{xj} and q_{yj} represent the components of the heat flux in the x and y directions, respectively, and k_j are the heat conductivities.

A. All Singularities Are in the Matrix

We consider now a circular elastic inclusion perfectly bonded to an infinite matrix subjected to a thermal field whose sources are in

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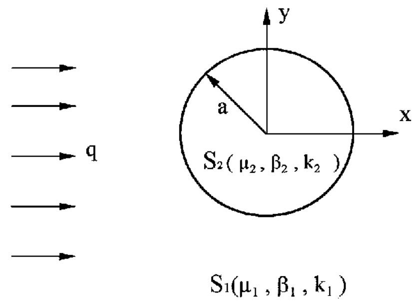

Fig. 1 Circular inclusion in an isotropic thermoelastic medium.

the matrix (including infinity) so that the thermal field is free of singularities inside or on the boundary of the circular inclusion. The proposed solution can be expressed as

$$g_1'(z) = g_0'(z) + \boldsymbol{g}_1'(z) \tag{3}$$

$$g_2'(z) = \boldsymbol{g}_2'(z) \tag{4}$$

where $g_0'(z)$ is the temperature function associated with the unperturbed field, whereas $\boldsymbol{g}_1'(z)$ [or $\boldsymbol{g}_2'(z)$] is the temperature function associated with the perturbed field of matrix (or inclusion). Because the matrix and the inclusion are assumed to be perfectly bonded along the interface, both the temperature and resultant heat flow must be continuous across the interface $z = \sigma = ae^{i\theta}$, resulting in

$$\left[g_0'^+(\sigma) + \boldsymbol{g}_1'^+(\sigma)\right] + \left[\overline{g_0'^+(\sigma)} + \overline{\boldsymbol{g}_1'^+(\sigma)}\right] = \left[\boldsymbol{g}_2'^-(\sigma) + \overline{\boldsymbol{g}_2'^-(\sigma)}\right] \tag{5}$$

$$ik_1\left\{\left[g_0'^+(\sigma) + \boldsymbol{g}_1'^+(\sigma)\right] - \left[\overline{g_0'^+(\sigma)} + \overline{\boldsymbol{g}_1'^+(\sigma)}\right]\right\} = ik_2\left[\boldsymbol{g}_2'^-(\sigma) - \overline{\boldsymbol{g}_2'^-(\sigma)}\right] \tag{6}$$

where an overbar denotes the complex conjugate and the superscript + (or −) is used for the boundary value approached from S_1 (or S_2). By using the property

$$\overline{\omega^+(\sigma)} = \lim_{z\to\sigma, z\in S_1} \overline{\omega(z)} = \lim_{z\to\sigma, z\in S_2} \overline{\omega(a^2/\bar{z})} = \overline{\omega}^-(a^2/\sigma) \tag{7}$$

Eqs. (5) and (6) are equivalent to the following relations:

$$\lim_{z\to\sigma, z\in S_1}\left\{\boldsymbol{g}_1'(z) - \overline{\boldsymbol{g}_2'}(a^2/z) + \overline{g_0'}(a^2/z)\right\} = \lim_{z\to\sigma, z\in S_2}\left\{\boldsymbol{g}_2'(z) - \overline{\boldsymbol{g}_1'}(a^2/z) - g_0'(z)\right\} \tag{8}$$

$$\lim_{z\to\sigma, z\in S_1}\left\{ik_1\boldsymbol{g}_1'(z) + ik_2\overline{\boldsymbol{g}_2'}(a^2/z) - ik_1\overline{g_0'}(a^2/z)\right\} = \lim_{z\to\sigma, z\in S_2}\left\{ik_2\boldsymbol{g}_2'(z) + ik_1\overline{\boldsymbol{g}_1'}(a^2/z) - ik_1 g_0'(z)\right\} \tag{9}$$

Based on the property of analytical continuation that if the function $f(z)$ is holomorphic in S_1 (or S_2) then the function $\bar{f}(a^2/z)$ is holomorphic in S_2 (or S_1), Eqs. (8) and (9) allow us to introduce a new set of complex potentials $\theta_j(z)$, $j = 1, 2$, which is holomorphic in the entire domain including the interface as

$$\theta_1(z) = \boldsymbol{g}_1'(z) - \overline{\boldsymbol{g}_2'}(a^2/z) + \overline{g_0'}(a^2/z) \tag{10}$$

$$\theta_2(z) = ik_1\boldsymbol{g}_1'(z) + ik_2\overline{\boldsymbol{g}}_2'(a^2/z) - ik_1\overline{g_0'}(a^2/z) \tag{11}$$

for $z \in S_1$ and

$$\theta_1(z) = \boldsymbol{g}_2'(z) - \overline{\boldsymbol{g}_1'}(a^2/z) - g_0'(z) \tag{12}$$

$$\theta_2(z) = ik_2\boldsymbol{g}_2'(z) + ik_1\overline{\boldsymbol{g}_1'}(a^2/z) - ik_1 g_0'(z) \tag{13}$$

for $z \in S_2$. Because $\theta_j(z)$ are now holomorphic and single valued in the whole plane including the point at infinity, and using Liouville's theorem, $\theta_j(z)$ is found to be a constant. However, the constant functions are treated as a reference temperature and can be assumed to be zero without loss in generality. With this result, the final expression of the temperature functions becomes

$$g_1'(z) = g_0'(z) + \frac{k_1 - k_2}{k_1 + k_2}\overline{g_0'}\left(\frac{a^2}{z}\right) \tag{14}$$

$$g_2'(z) = \frac{2k_1}{k_1 + k_2} g_0'(z) \tag{15}$$

B. All Singularities Are in the Inclusion

In this section we consider a circular inclusion perfectly bonded to an infinite matrix subjected to a thermal field whose sources are inside the inclusion. The proposed solution is given by

$$g_1'(z) = \boldsymbol{g}_1'(z) \tag{16}$$

$$g_2'(z) = g_0'(z) + \boldsymbol{g}_2'(z) \tag{17}$$

By applying the interface continuity conditions and the method of analytical continuation, similar to the preceding approach, the final expression of the temperature functions becomes

$$g_1'(z) = \frac{2k_2}{k_1 + k_2} g_0'(z) \tag{18}$$

$$g_2'(z) = g_0'(z) + \frac{k_2 - k_1}{k_1 + k_2}\overline{g_0'}\left(\frac{a^2}{z}\right) \tag{19}$$

Stress Field

The basic equations for the two-dimensional theory of thermoelasticity can be expressed as

$$2(\lambda_j + \mu_j)\frac{\partial}{\partial\bar{z}}\left(\frac{\partial D_j}{\partial z} + \frac{\overline{\partial D_j}}{\partial z}\right) + 4\mu_j\frac{\partial^2 D_j}{\partial\bar{z}\partial z} = 0, \qquad j = 1, 2$$

where λ_j and μ_j are the Lamé constants and $D_j = u_j + iv_j$ is the displacement. The boundary conditions used in the present problem are defined as $D_1 = D_2$ and $-Y_1 + iX_1 = -Y_2 + iX_2$ along the interface $|z| = a$, where the displacement $u_j + iv_j$ and resultant force $-Y_j + iX_j$ for each material medium can be expressed in terms of the complex potentials $\phi_j(z)$ and $\psi_j(z)$ as

$$2\mu_j(u_j + iv_j) = \kappa_j\phi_j(z) - z\overline{\phi_j'(z)} - \overline{\psi_j(z)} + 2\mu_j\beta_j g_j(z) \tag{20}$$

$$-Y_j + iX_j = \phi_j(z) + z\overline{\phi_j'(z)} + \overline{\psi_j(z)} \tag{21}$$

where μ_j is the shear modulus, $\kappa_j = 3 - \nu_j/1 + \nu_j$, $\beta_j = \alpha_j$ for plane stress and $\kappa_j = 3 - 4\nu_j$, $\beta_j = (1 + \nu_j)\alpha_j$ for plane strain, where ν_j is Poisson's ratio and α_j are the thermal expansion coefficients. The components of stress in polar coordinates are related to $\phi_j(z)$ and $\psi_j(z)$ by

$$[\sigma_{rr} + \sigma_{\theta\theta}]_j = 4\,\mathrm{Re}[\phi_j'(z)] \tag{22}$$

$$[\sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta}]_j = 2[\bar{z}\phi_j''(z) + \psi_j'(z)]\,e^{2i\theta} \tag{23}$$

A. All Singularities Are in the Matrix

We consider a circular inclusion perfectly bonded to an infinite matrix subjected to an elastic field whose sources are in the matrix. The proposed solution can be expressed as[14]

$$\phi_1(z) = \phi_0(z) + \boldsymbol{\phi}_1(z), \qquad \psi_1(z) = \psi_0(z) + \boldsymbol{\psi}_1(z) \tag{24}$$

$$\phi_2(z) = \boldsymbol{\phi}_2(z), \qquad \psi_2(z) = \boldsymbol{\psi}_2(z) \tag{25}$$

where $\phi_0(z)$ and $\psi_0(z)$ are the stress functions associated with the unperturbed field, whereas $\boldsymbol{\phi}_1(z)$, $\boldsymbol{\psi}_1(z)$ [or $\boldsymbol{\phi}_2(z)$, $\boldsymbol{\psi}_2(z)$] are the functions corresponding to the perturbed field of the matrix (or inclusion). Because the matrix and the inclusion are assumed to be perfectly bonded along the common boundary, both the displacement and traction force must be continuous across the interface $z = \sigma = ae^{i\theta}$, resulting in

$$\phi_0^+(\sigma) + \phi_1^+(\sigma) + \sigma\overline{\phi_0'^+(\sigma)} + \sigma\overline{\phi_1'^+(\sigma)} + \overline{\psi_0^+(\sigma)} + \overline{\psi_1^+(\sigma)} = \phi_2^-(\sigma) + \sigma\overline{\phi_2'^-(\sigma)} + \overline{\psi_2^-(\sigma)} \tag{26}$$

$$\frac{1}{2\mu_1}\left\{\kappa_1\left[\phi_0^+(\sigma) + \phi_1^+(\sigma)\right] - \sigma\overline{\phi_0'^+(\sigma)} - \sigma\overline{\phi_1'^+(\sigma)} - \overline{\psi_0'^+(\sigma)} - \overline{\psi_1^+(\sigma)}\right\} + \beta_1 g_0^+(\sigma) + \frac{(k_1 - k_2)\beta_1}{k_1 + k_2}\overline{f_0^+}\left(\frac{a^2}{\sigma}\right) = \frac{1}{2\mu_2}\left\{\kappa_2\phi_2^-(\sigma) - \sigma\overline{\phi_2'^-(\sigma)} - \overline{\psi_2^-(\sigma)}\right\} + \frac{2k_1\beta_2}{k_1 + k_2}g_0^-(\sigma) \tag{27}$$

where

$$f_0(z) = -a^2\int g_0'(z)\frac{dz}{z^2}$$

By using the property defined from Eq. (7), Eqs. (26) and (27) are equivalent to the following relations:

$$\lim_{z\to\sigma, z\in S_1}\left\{\phi_1(z) + z\overline{\phi_0'}(a^2/z) + \overline{\psi_0}(a^2/z) - z\overline{\phi_2'}(a^2/z) - \overline{\psi_2}(a^2/z)\right\} = \lim_{z\to\sigma, z\in S_2}\left\{\phi_2(z) - \phi_0(z) - z\overline{\phi_1'}(a^2/z) - \psi_1(a^2/z)\right\} \tag{28}$$

$$\lim_{z\to\sigma, z\in S_1}\left\{\frac{\kappa_1}{2\mu_1}\phi_1(z) - \frac{1}{2\mu_1}\left[z\overline{\phi_0'}\left(\frac{a^2}{z}\right) + \overline{\psi_0}\left(\frac{a^2}{z}\right)\right] + \frac{1}{2\mu_2}\left[z\overline{\phi_2'}\left(\frac{a^2}{z}\right) + \overline{\psi_2}\left(\frac{a^2}{z}\right)\right] + \frac{(k_1 - k_2)\beta_1}{k_1 + k_2}\overline{f_0}\left(\frac{a^2}{z}\right)\right\}$$
$$= \lim_{z\to\sigma, z\in S_2}\left\{\frac{\kappa_2}{2\mu_2}\phi_2(z) + \frac{1}{2\mu_1}\left[z\overline{\phi_1'}\left(\frac{a^2}{z}\right) + \overline{\psi_1}\left(\frac{a^2}{z}\right) - \kappa_1\phi_0(z)\right] + \left(\frac{2k_1}{k_1 + k_2}\beta_2 - \beta_1\right)g_0(z)\right\} \tag{29}$$

Based on the method of analytical continuation, Eqs. (28) and (29) allow us to introduce a new set of complex potentials $\vartheta_j(z)$, which are holomorphic in the entire domain including the interface boundary, defined as

$$\vartheta_1(z) = \phi_1(z) + z\overline{\phi_0'}(a^2/z) - z\overline{\phi_0'(0)} + \overline{\psi_0}(a^2/z) - z\overline{\phi_2'}(a^2/z) + z\overline{\phi_2'(0)} - \overline{\psi_2}(a^2/z) \tag{30}$$

$$\vartheta_2(z) = \frac{\kappa_1}{2\mu_1}\phi_1(z) - \frac{1}{2\mu_1}\left[z\overline{\phi_0'}\left(\frac{a^2}{z}\right) - z\overline{\phi_0'(0)} + \overline{\psi_0}\left(\frac{a^2}{z}\right)\right] + \frac{1}{2\mu_2}\left[z\overline{\phi_2'}\left(\frac{a^2}{z}\right) - z\overline{\phi_2'(0)} + \overline{\psi_2}\left(\frac{a^2}{z}\right)\right] + \frac{(k_1 - k_2)\beta_1}{k_1 + k_2}\overline{f_0}\left(\frac{a^2}{z}\right) \tag{31}$$

for $z \in S_1$ and

$$\vartheta_1(z) = \phi_2(z) - \phi_0(z) - z\overline{\phi_1'}(a^2/z) - z\overline{\phi_0'(0)} - \psi_1(a^2/z) + z\overline{\phi_2'(0)} \tag{32}$$

$$\vartheta_2(z) = \frac{\kappa}{2\mu_2}\phi_2(z) - \frac{1}{2\mu_2}z\overline{\phi_2'(0)} + \frac{1}{2\mu_1}\left[z\overline{\phi_1'}\frac{a^2}{z} + z\overline{\phi_0'(0)} + \overline{\psi_1}\left(\frac{a^2}{z}\right) - \kappa_1\phi_0(z)\right] + \left(\frac{2k_1}{k_1 + k_2}\beta_2 - \beta_1\right)g_0(z) \tag{33}$$

for $z \in S_2$. Note that the terms $z\overline{\phi_0'(0)}$ and $z\overline{\phi_2'(0)}$, which would become unbounded as $z \to \infty$, have been subtracted from the complex potentials $z\overline{\phi_0'}(a^2/z)$ and $z\overline{\phi_2'}(a^2/z)$, respectively, as indicated in Eqs. (30) and (31). Now $\vartheta_j(z)$ are holomorphic and single valued in the whole plane, including the point at infinity, and by Liouville's theorem, $\vartheta_j(z) \equiv$ const. However, the constant functions correspond to a rigid-body motion, which may be neglected. With this result, the final expression of the stress functions becomes

$$\phi_1(z) = \phi_0(z) + \gamma_3\left[z\overline{\phi_0'}(a^2/z) - z\overline{\phi_0'(0)} + \overline{\psi_0}(a^2/z)\right] + \gamma_2\overline{f_0}(a^2/z) \tag{34}$$

$$\psi_1(z) = \psi_0(z) + \gamma_1\overline{\phi_0}\left(\frac{a^2}{z}\right) + \gamma_3\frac{a^4}{z^3}\left[\overline{\psi_0'}\left(\frac{a^2}{z}\right) + z\overline{\phi_0''}\left(\frac{a^2}{z}\right) - \frac{z^2}{a^2}\overline{\phi_0'}\left(\frac{a^2}{z}\right)\right] + \gamma_4\overline{g_0}\left(\frac{a^2}{z}\right) + \frac{a^4}{z^3}\gamma_2\overline{f_0'}\left(\frac{a^2}{z}\right)$$
$$+ \frac{a^2}{z}\left\{\left[\frac{1+\gamma_1}{1-\gamma_3^*}\gamma_3^* + \gamma_3\right]\overline{\phi_0'(0)} + \left[\frac{1+\gamma_1}{1-\gamma_3^*} - 1\right]\phi_0'(0) + \frac{\gamma_4}{1-\gamma_3^*}\left[\gamma_3^*\overline{g_0'(0)} + g_0'(0)\right]\right\} \tag{35}$$

$$\phi_2(z) = (1+\gamma_1)\phi_0(z) + \gamma_4 g_0(z) + \frac{\gamma_3^*}{1-\gamma_3^{*2}}z\left\{(1+\gamma_1)\times\left[\gamma_3^*\phi_0'(0) + \overline{\phi_0'(0)}\right] + \gamma_4\left[\gamma_3^* g_0'(0) + \overline{g_0'(0)}\right]\right\} \tag{36}$$

$$\psi_2(z) = (1+\gamma_3)\left[(a^2/z)\phi_0'(z) - (a^2/z)\phi_0'(0) + \psi_0(z)\right] - (a^2/z)\phi_2'(z) + \gamma_2 f_0(z) + (a^2/z)\phi_2'(0) \tag{37}$$

where

$$\gamma_1 = \frac{\kappa_1\mu_2 - \kappa_2\mu_1}{\kappa_2\mu_1 + \mu_2}, \qquad \gamma_2 = \frac{2\mu_1\mu_2\beta_1}{\kappa_1\mu_2 + \mu_1}\frac{k_2 - k_1}{k_1 + k_2}$$
$$\gamma_3 = \frac{\mu_2 - \mu_1}{\kappa_1\mu_2 + \mu_1}, \qquad \gamma_4 = \frac{2\mu_1\mu_2}{\kappa_2\mu_1 + \mu_2}\left(\beta_1 - \frac{2k_1}{k_1 + k_2}\beta_2\right) \tag{38}$$
$$\gamma_3^* = \frac{\mu_1 - \mu_2}{\kappa_2\mu_1 + \mu_2}$$

If one assumes $\gamma_2 = \gamma_4 = 0$ and $\phi_0'(0) = \overline{\phi_0'(0)}$, Eqs. (34–37) can be simplified to the results provided by Honein et al.[11] for the corresponding isothermal problem.

B. All Singularities Are in the Inclusion

If all singularities are located inside the inclusion that is perfectly bonded to an infinite matrix, the stress functions can be expressed as[14]

$$\phi_1(z) = \phi_1(z), \qquad \psi_1(z) = \psi_1(z) \tag{39}$$

$$\phi_2(z) = \phi_0(z) + \phi_2(z), \qquad \psi_2(z) = \psi_0(z) + \psi_2(z) \tag{40}$$

Similar to the earlier approach, the stress functions associated with the perturbed field can be obtained using the continuity conditions and the method of continuation theorem. The stress functions finally become

$$\phi_1(z) = \left(1+\gamma_1^*\right)\phi_0(z) + \gamma_4^* g_0(z) \tag{41}$$

$$\psi_1(z) = \left(1+\gamma_1^*\right)\psi_0(z) + \left(\gamma_3^* - \gamma_1^*\right)\left[\psi_0(z) + \frac{a^2}{z}\phi_0'(z)\right] + \frac{a^2}{z}\frac{1}{1-\gamma_3^*}\left[\gamma_3^{*2}\overline{(\hat{\psi}_0)'(0)} + \gamma_3^*(\hat{\psi}_0)'(0) + \gamma_2^*\gamma_3^*\overline{(\hat{f}_0)'(0)} + \gamma_2^*(\hat{f}_0)'(0)\right] + \gamma_2^* g_0(z) - \frac{a^2}{z}\gamma_4^* g_0'(z) \tag{42}$$

$$\phi_2(z) = \phi_0(z) + \gamma_3^*\left[\overline{\psi_0}\left(\frac{a^2}{z}\right) + z\overline{\phi_0'}\left(\frac{a^2}{z}\right)\right] + \frac{\gamma_3^* z}{1-\gamma_3^{*2}}$$
$$\times\left[\gamma_3^{*2}(\hat{\psi}_0)'(0) + \gamma_3^*\overline{(\hat{\psi}_0)'(0)} + \gamma_2^*\gamma_3^*(\hat{f}_0)'(0) + \gamma_2^*\overline{(\hat{f}_0)'(0)}\right] + \gamma_2^*\hat{f}_0\left(\frac{a^2}{z}\right) \tag{43}$$

$$\psi_2(z) = \psi_0(z) + \gamma_1^*\overline{\phi_0}(a^2/z) + \gamma_3^*(a^4/z^3)\left[\overline{\psi_0'}(a^2/z) - (z^2/a^2)\overline{\phi_0'}(a^2/z) + z\overline{\phi_0''}(a^2/z)\right] + (a^2/z)\left[\gamma_3^*(\hat{\psi}_0)'(0) + \gamma_2^*(\hat{f}_0)'(0)\right] + \gamma_4^*\overline{g_0}(a^2/z) + \gamma_2^*(a^4/z^3)\overline{f_0'}(a^2/z) \tag{44}$$

where

$$\hat{\psi}_0(z) = \overline{\psi_0}(a^2/z), \qquad \hat{f}_0(z) = \overline{f_0}(a^2/z)$$

and the constants $\gamma_1^*, \gamma_2^*, \gamma_3^*$, and γ_4^* are obtained from $\gamma_1, \gamma_2, \gamma_3$, and γ_4, respectively, by interchanging the material properties of matrix and inclusion. If all terms concerning thermal effects such as $g_0(z)$ and $\overline{f_0}(z)$ are removed, Eqs. (41–44) are exactly the same as those for the corresponding isothermal problem provided by Honein et al.[11]

Formulation of Integral Equations for an Insulated Hole

In this section we first consider a single hole L, which is embedded in a matrix, interacting with an elastic circular inclusion. The corresponding homogeneous solutions associated with a single hole can be represented by distributed temperature dislocations and edge dislocations along the hole border as[15]

$$g_0'(z) = -\frac{i}{2\pi}\int_L b_0(s)\log(z-t)\,\mathrm{d}s \tag{45}$$

$$\phi_0(z) = \frac{i\mu_1}{\pi(1+\kappa_1)}\int_L [b_1(s) + ib_2(s)]\log(z-t)\,\mathrm{d}s \tag{46}$$

$$\psi_0(z) = \frac{-i\mu_1}{\pi(1+\kappa_1)}\int_L [b_1(s) - ib_2(s)]\log(z-t)\,\mathrm{d}s - \frac{i\mu_1}{\pi(1+\kappa_1)}\int_L \frac{[b_1(s)+ib_2(s)]\bar{t}}{z-t}\,\mathrm{d}s \tag{47}$$

where $b_0(s)$ is the strength of the temperature dislocation and $b_1(s)$ and $b_2(s)$ are the components of the displacement discontinuities across the dislocation line. The temperature function $b_0(s)$ can be found from the thermal boundary condition of an insulated hole such that the total heat flux across the hole surface must be balanced by the given resultant heat flux Q_1 across the hole border L in the unflawed media, i.e.,

$$Q_1 = -k_1\,\mathrm{Im}[g_1'(t)] + c_0, \qquad t \in L \tag{48}$$

where c_0 is a constant. In addition, the single-valued condition of the temperature must be satisfied, i.e.,

$$\int_L b_0(s)\,\mathrm{d}s = 0 \tag{49}$$

Substitution of Eq. (45) into Eq. (14) and application of Eqs. (48) and (49) results in the singular integral equation for solving the unknown function $b_0(s)$. On the other hand, the unknown functions $b_1(s)$ and $b_2(s)$ can be determined from the traction-free boundary condition such that the force acting on the hole surface must be balanced by the given resultant force applied on the hole border, i.e.,

$$-Y_1 + iX_1 = \phi_1(t) + t\overline{\phi_1'(t)} + \overline{\psi_1(t)} + c_1 + ic_2, \qquad t \in L \tag{50}$$

where c_1 and c_2 are real constants to be determined. Moreover, the requirement of single-valued displacements given by

$$\int_L [b_1(s) + ib_2(s)]\,\mathrm{d}s = \int_L \beta_1\left[\int b_0(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \tag{51}$$

must be satisfied. Substitution of Eqs. (46) and (47) into Eqs. (34) and (35) and application of Eqs. (50) and (51) yields the singular integral equation for solving the unknown functions $b_1(s)$ and $b_2(s)$.

Next we consider the problem of when a hole is embedded in the circular inclusion. The corresponding homogeneous solutions are obtained from Eqs. (45–47), except that the material properties μ_1 and κ_2 indicated in Eqs. (46) and (47) are replaced by μ_2 and κ_2, respectively. In a way similar to the earlier approach, the resulting singular integral equations can be established from Eqs. (48–51) by replacing the material constant k_1 in Eq. (48) and β_1 in Eq. (51) with k_2 and β_2, respectively, and the stress functions $\phi_1(z)$, $\psi_1(z)$ in Eq. (50) with $\phi_2(z)$, $\psi_2(z)$.

Numerical Results and Discussion

The dislocation functions $b_0(s)$, $b_1(s)$, and $b_2(s)$ appearing in the preceding singular integral equations will be solved numerically using the appropriate interpolation formulas. For the purpose of performing the numerical calculation, the contour L is replaced by a polygon of N line segments (Fig. 2). The interpolation formulas for line segments in local coordinates $s_j (1 \le j \le N)$ are taken as[15]

$$b_i(s_j) = b_{i,j}\frac{d_j - s_j}{2d_j} + b_{i,j+1}\frac{d_j + s_j}{2d_j} \qquad (i = 0, 1, 2) \tag{52}$$

where $d_j, 1 \le j \le N$, are the half-lengths for each line segment and $b_{i,j}, 0 \le j \le N$, are the unknown coefficients to be determined. If the preceding formulas are used, the boundary integral equation for the temperature function, Eq. (48), together with the subsidiary condition, Eq. (49), can be carried out to yield $N+2$ algebraic equations for solving $N+2$ unknown coefficients ($b_{0,0}, b_{0,1}, b_{0,2}, \ldots, b_{0,N}, c_0$). Similarly, the boundary integral equation for the stress functions, Eq. (50), together with the subsidiary condition, Eq. (51), can be arranged to yield $2N+4$ algebraic equations for solving $2N+4$ unknown coefficients ($b_{1,0}, b_{1,1}, \ldots, b_{1,N}, b_{2,0}, b_{2,1}, \ldots, b_{2,N}, c_1, c_2$). Once the stress functions are obtained, the hoop stresses along the hole boundary can be evaluated by

$$\sigma_{\theta\theta} = 4\,\mathrm{Re}[\phi'(z)] \tag{53}$$

A. Hole Embedded in a Matrix

As our first example, an insulated circular hole interacting with an elastic circular inclusion under a remote uniform heat flow is considered (Fig. 3). To perform the numerical technique, the contour of the circular hole is replaced by a polygon of N line elements discreted with a number of N points expressed by

$$x_i = a\cos\left[\frac{2(i-1)\pi}{N}\right], \qquad y_i = a\sin\left[\frac{2(i-1)\pi}{N}\right] \qquad (x_i, y_i) \in L \tag{54}$$

The calculated hoop stresses along the circular hole boundary with the number of line segments $N = 40$, which are checked to achieve a good accuracy with an error less than 1% as compared to those obtained by Chao et al.[13] with $d/a = 4$, are shown in Figs. 4–10. It is seen that, as a hole approaches the circular inclusion with the

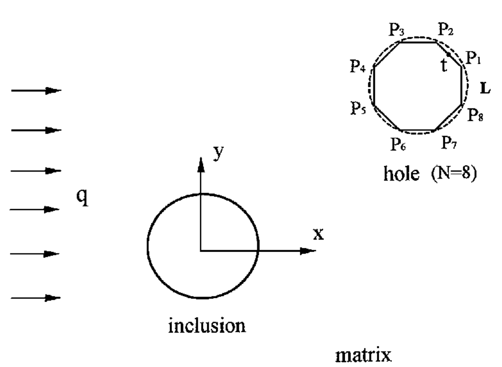

Fig. 2 Polygon of N line elements for simulating a circular hole.

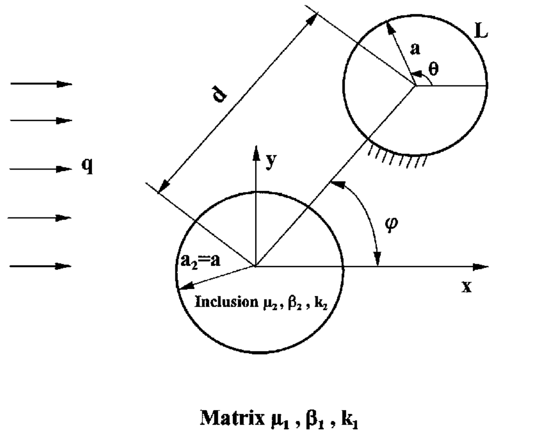

Fig. 3 Circular inclusion interacted with a circular hole located in the matrix.

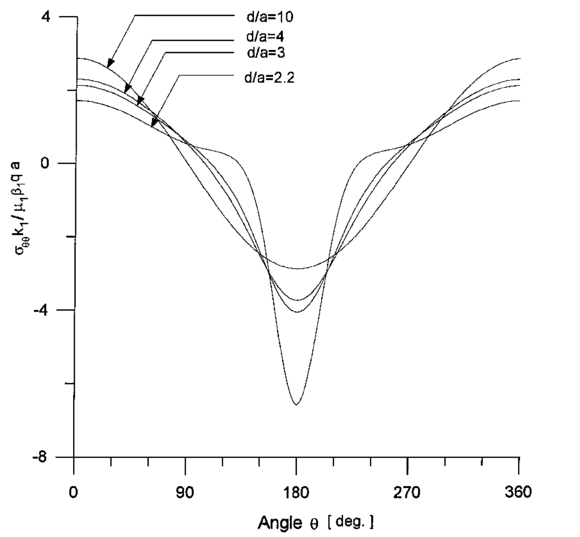

Fig. 4 Variation of hoop stress along a circular hole with $\varphi = 0$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$.

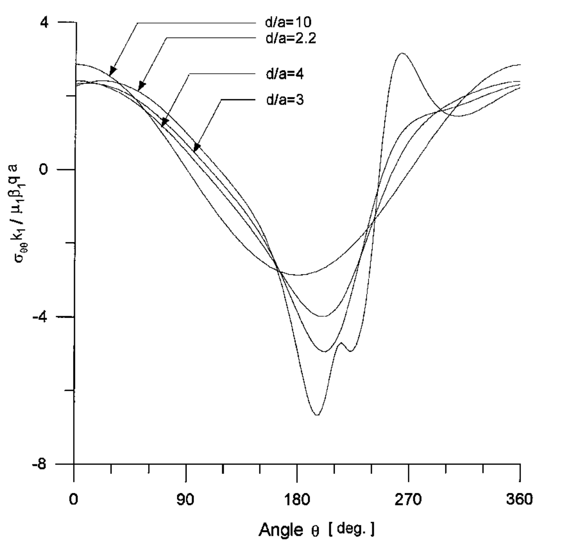

Fig. 5 Variation of hoop stress along a circular hole with $\varphi = 45$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$ ($d/a \geq 2.2$).

case $\phi = 0$ deg (or $\phi = 180$ deg), the maximum absolute hoop stress occurring at the point $\theta = 180$ deg (or $\theta = 0$ deg), which is nearest to the neighboring inclusion, is increased (Figs. 4 and 10). Note that, as a hole approaches the circular inclusion with the distance $d/a = 2.2$ for the case $\phi = 45$ deg (or $\phi = 135$ deg), the maximum absolute hoop stress takes place at the point $\theta = 200$ deg (or $\theta = 340$ deg), which is not exactly the point $\theta = 225$ deg (or $\theta = 315$ deg) nearest to the neighboring inclusion (Figs. 5 and 9). It is expected that, as

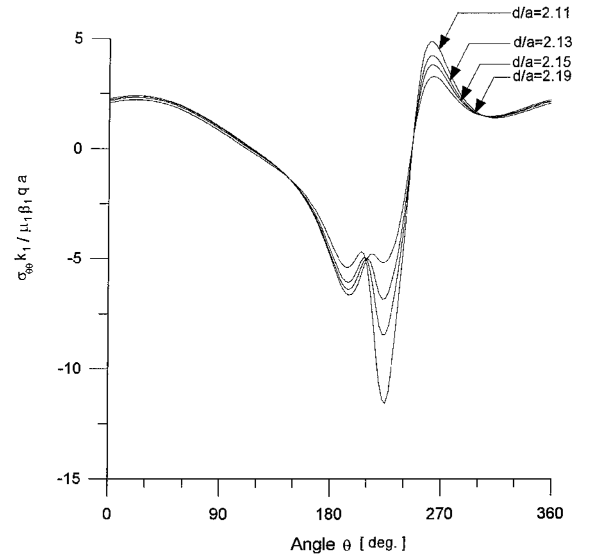

Fig. 6 Variation of hoop stress along a circular hole with $\varphi = 45$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$ ($d/a \leq 2.19$).

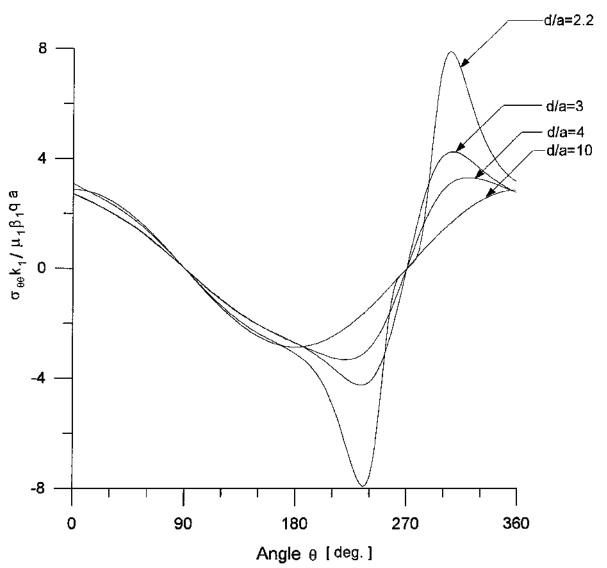

Fig. 7 Variation of hoop stress along a circular hole with $\varphi = 90$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$.

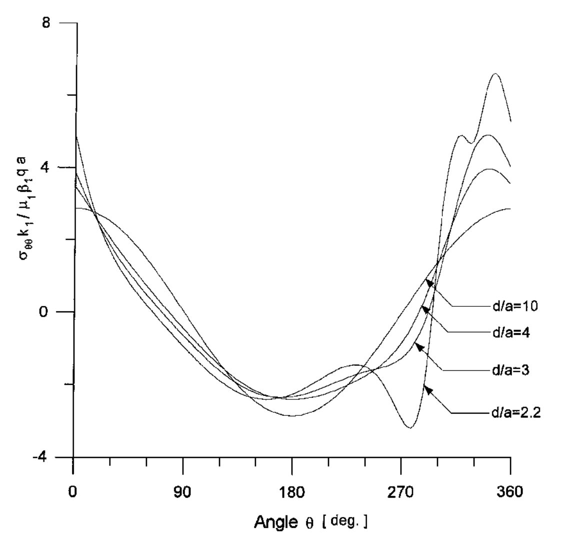

Fig. 8 Variation of hoop stress along a circular hole with $\varphi = 135$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$ ($d/a \geq 2.2$).

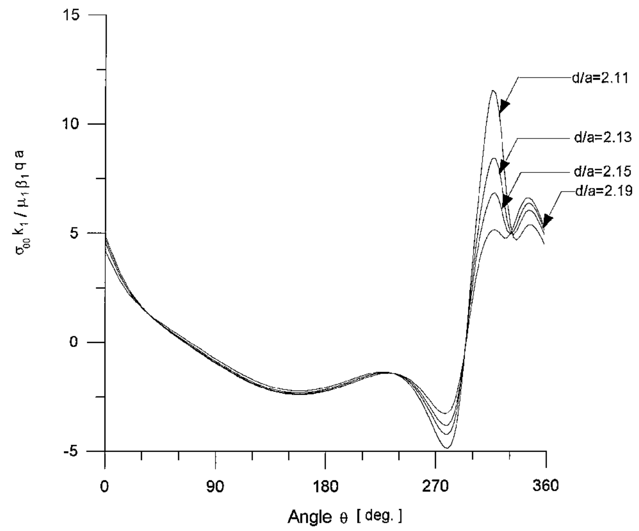

Fig. 9 Variation of hoop stress along a circular hole with $\varphi = 135$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$ ($d/a \leq 2.19$).

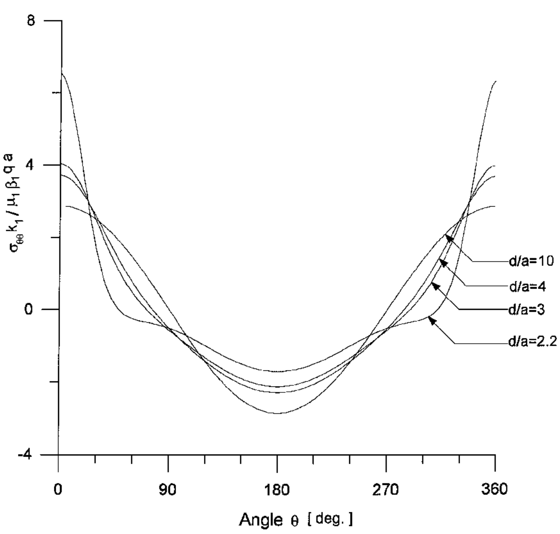

Fig. 10 Variation of hoop stress along a circular hole with $\varphi = 180$ deg and $\mu_2/\mu_1 = \beta_2/\beta_1 = k_2/k_1 = 0.1$.

a hole is infinitely close to the neighboring circular inclusion, for example, $d/a = 2.11$ as displayed in Figs. 6 and 8, the maximum hoop stress would occur at the point along a hole boundary that is closest to the neighboring inclusion. For the case $\phi = 90$ deg, i.e., a hole and the circular inclusion arrayed perpendicular to a remote uniform heat flow, the hoop stress is found to vanish at the points $\theta = 90$ and 270 deg no matter how close a hole is to the circular inclusion (see Fig. 7). When a hole is far away from the circular inclusion with the distance $d/a = 10$, the hoop stresses along a hole boundary are nearly the same as those of the corresponding single-hole problem and the presence of the circular inclusion can be completely ignored. Keep in mind that the positive (or negative) thermal stress is always accompanied by the presence of lower temperature (or higher temperature). The effect of material properties on the local stress along the hole boundary is shown in Fig. 11. The results show that, for a fixed value of $d/a = 4$ and $\phi = 90$ deg, the maximum negative (or positive) hoop stress takes place at the point $\theta = 225$ deg (or $\theta = 315$ deg) along the hole border as the neighboring elastic inclusion becomes a hole. Furthermore, the hoop stress vanishes at the points $\theta = 90$ and 270 deg regardless of the rigidity of the circular inclusion.

B. Hole Embedded in the Inclusion

As our second example we consider the heterogeneous problem with a hole embedded in the inclusion under a remote uniform heat flow (Fig. 12). The effect of heat conductivity and thermal expansion coefficient on the local stress along a hole boundary is shown in

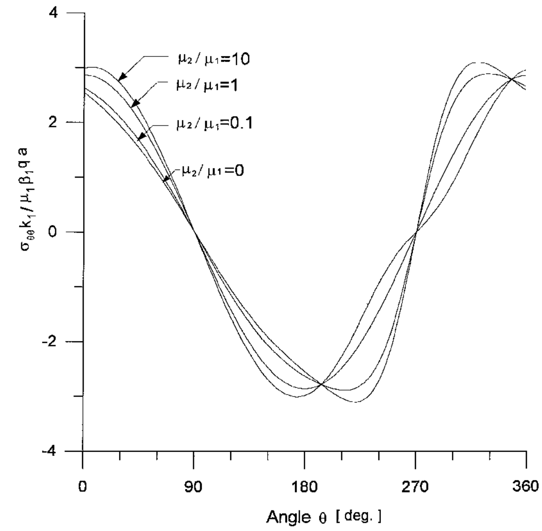

Fig. 11 Variation of hoop stress along a circular hole with μ_2/μ_1 by fixing $\varphi = 90$ deg, $d/a = 4$, and $\beta_2/\beta_1 = k_2/k_1 = 1$.

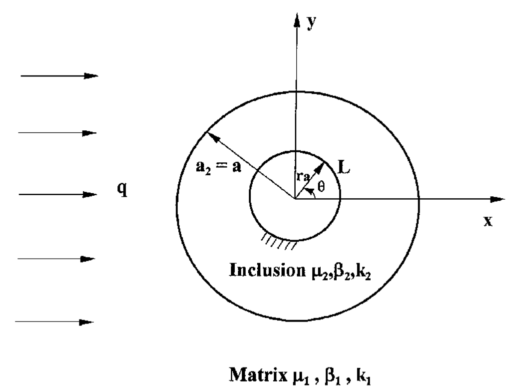

Fig. 12 Circular hole located in the inclusion embedded in an infinite matrix.

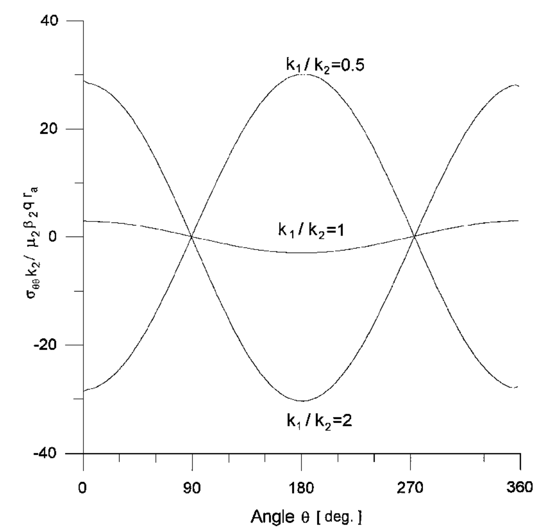

Fig. 13 Variation of hoop stress along a circular hole with k_1/k_2, by fixing $r_a/a = 0.1$ and $\mu_2/\mu_1 = \beta_2/\beta_1 = 1$.

Figs. 13 and 14, respectively. It is interesting to note that the hoop stress may change sign if the inclusion having the higher (or lower) conductivity is replaced by the one having the lower (or higher) conductivity, as indicated in Fig. 14. This conclusion can be further justified by the expression of

$$\lambda_2^* = \frac{2\mu_2\mu_1\beta_2}{\kappa_2\mu_1 + \mu_2}\frac{k_1 - k_2}{k_2 + k_1}$$

as indicated in Eq. (43), which is found to change sign as $k_1/k_2 > 1$ (or $k_1/k_2 < 1$) is replaced by $k_1/k_2 < 1$ (or $k_1/k_2 > 1$). For a special

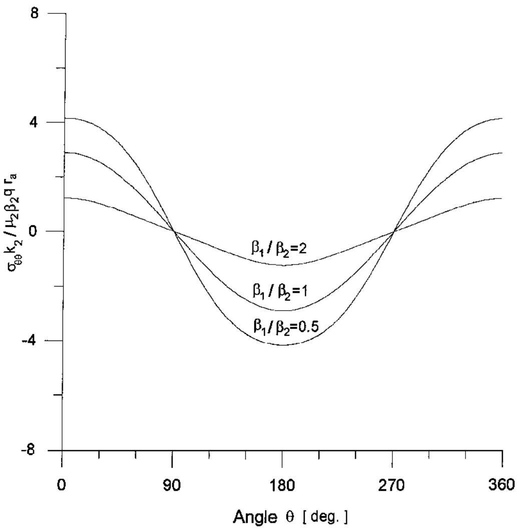

Fig. 14 Variation of hoop stress along a circular hole with β_1/β_2, by fixing $r_a/a = 0.1$ and $\mu_2/\mu_1 = k_2/k_1 = 1$.

case with $k_1 = k_2$, the distribution of hoop stress along a hole boundary embedded in an inclusion is found to be exactly the same as that of the corresponding single-hole problem. From Fig. 14, we may conclude that the magnitude of the absolute hoop stress will be increased if the thermal expansion coefficient of the inclusion becomes larger as compared to that of the matrix.

Concluding Remarks

The interaction between an arbitrarily located hole and an elastic circular inclusion under a remote uniform heat flow is investigated by observing the hoop stress along a hole boundary. By combining Green's function as derived in this paper and the existing solutions for dislocation functions, the singular integral equations for the thermoelastic problem are formulated and the unknown coefficients, which are related to the magnitude of hoop stress, are solved numerically by applying the appropriate interpolation formulas. Some significant conclusions that are drawn are the following: 1) The maximum absolute hoop stress is always found to occur at the point that is nearest to the neighboring inclusion when a hole and the circular inclusion are arrayed parallel to a remote uniform heat flow, i.e., $\phi = 0$ and 180 deg. 2) For the other cases, with $\phi \neq 0$ and 180 deg, the maximum absolute hoop stress would tend to occur at the point that is nearest to the neighboring inclusion as a hole is infinitely close to the neighboring inclusion. 3) As a hole is embedded in the inclusion, the hoop stress may change sign if one substitutes the inclusion having the higher (or lower) heat conductivity with the one having the lower (or higher) heat conductivity.

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